

# THE $(2, 3)$ -GENERATION OF THE SPECIAL UNITARY GROUPS OF DIMENSION 6

M.A. PELLEGRINI, M. PRANDELLI, AND M.C. TAMBURINI BELLANI

**ABSTRACT.** In this paper we give explicit  $(2, 3)$ -generators of the unitary groups  $SU_6(q^2)$ , for all  $q$ . They fit into a uniform sequence of likely  $(2, 3)$ -generators for all  $n \geq 6$ .

## 1. INTRODUCTION

A  $(2, 3)$ -generated group is a group that can be generated by two elements of respective orders 2 and 3. Apart from the infinite families  $PSp_4(2^a)$  and  $PSp_4(3^a)$ , the other finite classical simple groups are  $(2, 3)$ -generated, up to a finite number of exceptions. This fact, proved in [4] with probabilistic methods, was suggested by the constructive positive results obtained, with permutational methods, for classical groups of sufficiently large rank (e.g. see [11], [2], [13], [14]). A natural question is to detect all the exceptions (Problem 18.49 in [5]). For  $n \leq 5$  the complete list of exceptions is:  $PSL_2(9)$ ,  $PSL_3(4)$ ,  $PSL_4(2)$ ,  $PSU_3(q^2)$  for  $q = 2, 3, 5$ ,  $PSU_4(q^2)$  for  $q = 2, 3$ , and  $PSU_5(4)$ . Moreover  $P\Omega_8^+(2)$  and  $P\Omega_8^+(3)$  are not  $(2, 3)$ -generated (M. Vsemirnov, 2012). According to what conjectured in [15], this should be the complete list.

The most obvious step towards the confirmation of this claim is to consider the cases  $n = 6, 7$ . The following groups are known to be  $(2, 3)$ -generated:

- $PSL_6(q)$  for all  $q$  (see [2] for  $p \neq 2$ ,  $q \neq 9$  and [10] for all  $q$ );
- $PSp_6(q)$  for all  $q$  (see [12] for  $q$  odd and [6] for  $q$  even);
- $PSL_7(q)$  for all  $q$  (see [2] for  $p \neq 2$ ,  $q \neq 9$  and [9] for all  $q$ );
- $PSU_7(q^2)$  and  $\Omega_7(q)$  for all  $q$  (see [6]);

From Proposition 2.9 and Theorem 2.10 of this paper follows Theorem 1.1, which closes the only case left open for  $n \leq 7$ , namely the 6-dimensional unitary groups.

**Theorem 1.1.** *The groups  $SU_6(q^2)$  and  $PSU_6(q^2)$  are  $(2, 3)$ -generated for all  $q$ .*

Noting that  $SU_n(4)$  is not  $(2, 3)$ -generated for  $3 \leq n < 6$ , the value  $n = 6$  seems a good lower bound for the existence of uniform  $(2, 3)$ -generators for the unitary groups. Indeed our  $(2, 3)$ -generators  $x = x_6$ ,  $y = y_6$  of  $SU_6(q^2)$ , described in (1) for  $q > 2$ , in (5) for  $q = 2$ , fit into a sequence of  $(2, 3)$ -elements  $x_n, y_n \in SU_n(q^2)$ ,  $n \geq 6$  (see the last Section). These elements, for  $q > 2$ , depend on a parameter  $a \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . Computer evidences, combined with the permutational methods mentioned above, strongly suggest that  $\langle x_n, y_n \rangle = SU_n(q^2)$ , whenever  $\langle x_6, y_6 \rangle = SU_6(q^2)$ . For  $q > 2$  a sufficient condition is that  $a$  satisfies the assumptions of Theorem 2.10.

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Throughout this paper,  $\mathbb{F}$  is an algebraically closed field of characteristic  $p \geq 0$ . The set  $\{e_1, e_2, \dots, e_n\}$  denotes the canonical basis of  $\mathbb{F}^n$ . When  $p > 0$  we have that  $\mathbb{F}_q \leq \mathbb{F}$  for any fixed power  $q$  of the prime  $p$ . Further, we denote by  $\sigma$  the automorphism of  $\mathrm{SL}_n(q^2)$  defined by  $(\alpha_{i,j}) \mapsto (\alpha_{i,j}^q)$ . Finally,  $\omega$  is an element of  $\mathbb{F}$  of order 3 if  $p \neq 3$ ,  $\omega = 1$  if  $p = 3$ .

## 2. THE GROUPS $\mathrm{SU}_6(q^2)$

In order to prove that the groups  $\mathrm{SU}_6(q^2)$  are  $(2, 3)$ -generated, we construct our generators. Let  $a \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  (in particular  $a \neq 0$ ) be such that  $a^{q+1} \neq 4$ . Also set:

$$\begin{aligned} b &= 2a - a^{2q}, & c &= a^{q+1} - 4, \\ d &= 2a^3 + 2a^{3q} - 12a^{q+1} + 16, & e &= -a^{4q} + 6a^{2q+1} - a^{q+3} - 8a^q. \end{aligned}$$

Consider now the subgroup  $H = \langle x, y \rangle$  of  $\mathrm{SL}_6(q^2)$ , where

$$(1) \quad x = x_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & a \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad y = y_6 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{b}{c} & \frac{e}{c^2} & \frac{d}{c^2} \\ 0 & 0 & 0 & 1 & -\frac{b}{c} & -\frac{b^q}{c} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Notice that

$$(2) \quad (a^3 - 8) = (a - 2)(a^2 + 2a + 4) \neq 0.$$

Namely, we are excluding  $a = 2$ , since  $a \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ . So  $a^3 = 8$  implies  $a = 2\omega^j$  and  $a^q = 2\omega^{2j}$  ( $j = 1, 2$ ). However, in this case  $c = a^{q+1} - 4 = 0$ , against our initial assumptions.

The similarity invariants of  $x, y$  are, respectively,

$$t - 1, t - 1, t^2 - 1, t^2 - 1 \quad \text{and} \quad t^3 - 1, t^3 - 1.$$

The characteristic polynomial of  $z = xy$  is

$$(3) \quad \chi_z(t) = t^6 - \frac{b+ac}{c}t^5 - \frac{b^q}{c}t^4 - \frac{b}{c}t^2 - \frac{b^q+ca^q}{c}t + 1.$$

**Lemma 2.1.** *If  $H = \langle x, y \rangle$  is absolutely irreducible, then the characteristic polynomial  $\chi_z(t)$  of  $z$  coincides with its minimal polynomial and  $H \leq \mathrm{SU}_6(q^2)$ .*

*Proof.* We have  $\dim C(x) = 16 + 4 = 20$ ,  $\dim C(y) = 4 + 4 + 4 = 12$ . By Scott's formula, when  $H$  is absolutely irreducible we get  $\dim C(z) \leq (6^2 + 2) - 20 - 12 = 6$ . From the Frobenius formula giving the dimension of centralizers of the rational canonical forms, we see that, for a fixed  $n$ , the minimal dimension is  $n$ , and it is attained precisely by the canonical forms having just one similarity invariant. It follows that  $z$  has a unique similarity invariant, whence our first claim. In particular, this means that the triple  $(x, y, z)$  is rigid and by [8, Theorem 3.1] we obtain  $H \leq \mathrm{SU}_6(q^2)$ .  $\square$

It is useful the Gram matrix of the hermitian form fixed by  $H$ , namely:

$$J = \frac{1}{c} \left( \begin{array}{c|cc} c^2 \cdot I_4 & 0 & \\ \hline 0 & d & e \\ & e^q & d \end{array} \right),$$

where  $J^T = J^\sigma$ ,  $x^T J x^\sigma = J$  and  $y^T J y^\sigma = J$ . Denoting

$$(4) \quad \gamma_j = a + \omega^{-j} a^q + 2\omega^j \quad (j = 0, 1, 2), \quad \gamma = a^3 + a^{3q} - 6a^{q+1} + 8 = \gamma_0 \gamma_1 \gamma_2,$$

we have  $\det(J) = -c^3 \gamma^2$ .

In the following proposition we make use of these subspaces of  $\mathbb{F}^6$ :

$$A = \langle e_3 - e_4, e_5 \rangle, \quad B = \langle e_3 - e_4, 2e_5 - ae_6 \rangle,$$

$$V_x = \langle e_1, e_2, e_3 + e_4, ae_5 + 2e_6 \rangle, \quad V_{x^T} = \langle e_1, e_2, e_3 + e_4, e_6 \rangle.$$

Notice that  $V_x$  and  $V_{x^T}$  are, respectively, the eigenspace of  $x$  and  $x^T$  associated to the eigenvalue 1. For  $p \neq 2$ ,  $A$  and  $B$  are, respectively, the eigenspace of  $x$  and  $x^T$  associated to the eigenvalue  $-1$ . For  $p = 2$ ,  $A \leq V_x$  and  $B \leq V_{x^T}$ .

**Proposition 2.2.** *Let  $a \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$  such that  $a^{q+1} \neq 4$ . The group  $H = \langle x, y \rangle$  is absolutely irreducible if, and only if, the following conditions hold:*

- (i)  $\det(J) = -c^3 \gamma^2 \neq 0$ , i.e.,  $a^3 - 6a^{q+1} + a^{3q} + 8 \neq 0$ ;
- (ii)  $a^{2q+2} - 5a^{q+1} + 8 \neq 0$ .

In particular for  $q = 2$  the group  $H$  is reducible over  $\mathbb{F}$ .

*Proof.* If  $\gamma = 0$  then the kernel of  $J$  is fixed by  $H$ . If  $\gamma \neq 0$  and  $a^{2q+2} - 5a^{q+1} + 8 = 0$ , then  $a^{2q} - a \neq 0$ . Indeed  $a^{2q} - a = 0$  would give  $a^q = a^2$ , whence  $a^{2q+2} - 5a^{q+1} + 8 = a^3 - 6a^{q+1} + a^{3q} + 8 = \gamma$ . In this case the subspace  $W$  generated by  $\{v, yv, y^2v\}$  with  $v = \left(0, 0, 1, -1, \frac{c}{a^{2q}-a}, 0\right)^T$  is  $H$ -invariant. So conditions (i) and (ii) are necessary. Now we show that they are also sufficient.

Assume that (i) and (ii) hold and let  $W$  be a proper  $H$ -invariant space.

**Case (a)**  $\dim(W) \leq 3$ .

**(a.1)** There exists  $0 \neq v \in A \cap W$ . Set  $v = (0, 0, x_1, -x_1, x_2, 0)^T$ .

We show that  $v, yv, y^2v$  are linearly independent. We consider the matrix  $A(i, j, k)$  with rows  $v[s], yv[s], y^2v[s]$  for  $s \in \{i, j, k\}$  and call  $\delta(i, j, k)$  its determinant. If  $x_1 = 0$ , we may assume  $x_2 = 1$ . Then  $\delta(4, 5, 6) = \frac{\gamma(a^{3q}-8)}{c^3}$  which is non-zero. In this case  $xyv \in W$  iff  $a^q \gamma = 0$ , which is an absurd. If  $x_1 \neq 0$ , then we may set  $x_1 = 1$ . In this case  $\delta(1, 2, 3) = 1$ . Thus  $\langle v, yv, y^2v \rangle$  has dimension 3 and coincides with  $W$ . We have that  $xyv \in W$  iff  $w_1 = xyv - yv = \lambda v$ , where  $\lambda$  is the coordinate of position 3 in  $w_1$ . Similarly  $xy^2v \in W$  iff  $w_2 = xy^2v - y^2v = \mu v$  where  $\mu$  is the coordinate of position 3 in  $w_2$ . Let  $p_1, p_2$  denote the coordinates of position 5 of  $w_1 - \lambda v$  and  $w_2 - \mu v$  respectively. Since all the remaining coordinates are 0, we have that  $W$  is  $H$ -invariant only if  $p_1 = p_2 = 0$ . In particular  $2p_1 + a^q p_2 = 0$  gives  $(a^{2q} - a)x_2 - c = 0$ . From the assumption  $c \neq 0$  it follows  $a^{2q} - a \neq 0$ , whence  $x_2 = \frac{c}{a^{2q}-a}$ . After this substitution direct calculation shows that  $p_1 = p_2 = 0$  iff  $a^{2q+2} - 5a^{q+1} + 8 = 0$ .

**(a.2)**  $A \cap W = 0$ .

Clearly, there exists  $0 \neq v \in V_1^x \cap W$ . Set  $v = (x_1, x_2, x_3, x_3, ax_4, 2x_4)^T$  and  $u_1 = yv - xyv$ ,  $u_2 = y^2v - xy^2v$ . Observe that  $u_1, u_2 \in A \cap W$ , hence  $u_1 = u_2 = 0$ . The coordinates of position 5 of  $u_1, u_2$  are, respectively,  $2x_3 + (2a^q - a^2)x_4$  and  $-ax_3 - cx_4$ . Considering the coordinate of position 5 of  $au_1 + 2u_2$  we get  $(a^3 - 8)x_4 = 0$ , whence  $x_4 = 0$ . After substituting of this value, the coordinate 5 of  $u_2$  becomes  $-ax_3$ , whence  $x_3 = 0$ . Thus we obtain  $u_1 = (0, 0, x_2, -x_2, 0, 0)^T$ ,  $u_2 = (0, 0, x_1, -x_1, 0, 0)^T$ . It follows  $x_1 = x_2 = 0$ , whence the contradiction  $v = 0$ .

**Case (b)**  $\dim(W) \geq 4$ .

In this case there exists an  $H^T$ -invariant space  $U$ , with  $\dim(U) \leq 2$ . Let  $0 \neq v \in B \cap U$  and set  $v = (0, 0, x_1, -x_1, 2x_2, -ax_2)^T$ . We have that  $v, y^T v, (y^T)^2 v$  must be linearly dependent. This easily implies  $x_1 = 0$ . After this substitution,  $(a^3 - 8)x_2^3 = 0$ , whence the contradiction  $x_2 = 0$ . It follows that  $B \cap U = 0$ . So there exists  $0 \neq v \in V_1^{x^T} \cap U$ . Set  $v = (x_1, x_2, x_3, x_3, 0, x_4)$  and  $u_1 = y^T v - x^T y^T v$ ,  $u_2 = (y^T)^2 v - x^T (y^T)^2 v$ . As above  $u_1, u_2 \in B \cap U$ , hence  $u_1 = u_2 = 0$ . Coordinates of position 4 give:  $-cx_1 + (2a - a^{2q})x_3 = 0$  and  $(2a^q - a^2)x_3 - cx_2 + cx_4 = 0$ . If  $x_3 = 0$  then, considering the other coordinates of  $u_1$  we get  $x_1 = x_4 = 0$ . In this case,  $u_2 = x_2(e_3 - e_4)$  and so  $x_2 = 0$  and  $v = 0$ . So assume  $x_3 = c$ . We get  $x_1 = 2a - a^{2q}$  and  $x_2 = 2a^q - a^2 + x_4$ . Coordinate 6 of  $u_1$  gives  $x_4 = \frac{a^{q+3} + a^{4q} - 6a^{2q+1} + 8a^q}{c}$ . Imposing that coordinate 6 of  $u_2$  is 0, we get  $a(a^{3q} - 8)\gamma = 0$ , a contradiction.  $\square$

**Lemma 2.3.** *If  $H$  is absolutely irreducible, then  $z^k$  is not scalar for  $k \leq 9$ .*

*Proof.* For  $k \leq 7$  our claim follows from the following observations:

$$\begin{aligned} ze_1 &= e_2, & z^4 e_1 &= z^5 e_3 = \sum_{j \neq 6} \lambda_j e_j - e_6, \\ z^2 e_1 &= z^3 e_3 = e_4, & z^6 e_1 &= z^7 e_3 = \sum_{j \neq 2} \mu_j e_j + \frac{a^q \gamma}{c^2} e_2. \end{aligned}$$

Now, assume  $z^8$  is scalar. From  $z^8 e_1 = \frac{\gamma\alpha}{c^4} e_2 + \frac{\gamma\beta}{c^3} e_4 + \sum_{j \neq 2,4} \lambda_j e_j$ , we get  $\alpha = \beta = 0$ , where

$$\begin{aligned} \alpha &= a^{3q+4} - 2a^{4q+2} - 7a^{2q+3} + a^{5q} + 8a^{3q+1} + 20a^{q+2} - 16a^{2q} - 16a, \\ \beta &= a^{2q+2} - a^{3q} - 4a^{q+1} + 8. \end{aligned}$$

Treating  $\alpha$  and  $\beta$  as polynomials in  $a$  and  $a^q$ , their resultant with respect to  $a^q$  is  $64(a^3 - 8)^3$ , whence  $p = 2$ . In this case  $\beta = a^{2q}(a^q + a^2)$ . Taking  $a^q = a^2$ , we obtain  $\lambda_1 = a^2 \neq 0$ . To prove that  $z^9$  is not scalar, simply note that  $z^9 e_3 = z^8 e_1$  and reapply the previous computations.  $\square$

We need to exclude that  $H$  is contained in any maximal subgroup of  $\text{SU}_6(q^2)$ . We refer to [1, Tables 8.26 and 8.27, pages 390-391] for the list of these subgroups.

**Lemma 2.4.** *If  $H$  is absolutely irreducible, then  $H$  is not monomial.*

*Proof.* Let  $\mathcal{B} = \{v_1, \dots\}$  be a basis of  $\mathbb{F}^6$  on which  $H$  acts monomially. Since  $H$  is irreducible, this action must be transitive. By this condition and the canonical form of  $x$  and  $y$ , we note that the permutation induced by  $x$  cannot be the product of three 2-cycles, and we may set:

$$yv_1 = v_2, yv_2 = v_3, xv_3 = v_4, yv_4 = v_5, yv_5 = v_6.$$

Thus  $\mathcal{B} = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ . Clearly  $yv_3 = v_1$ ,  $xv_4 = v_3$  and  $yv_6 = v_4$ . Considering the canonical form of  $x$ , only the following cases have to be analyzed.

**Case (a)** The permutation induced by  $x$  is a 2-cycle ( $p$  odd). Thus:

$$xv_i = \delta_i v_i, \quad (i = 1, 2, 5, 6), \quad \delta_i = \pm 1.$$

In this case  $(xy)^6$ , is scalar, a contradiction with Lemma 2.3.

**Case (b)** The permutation induced by  $x$  is the product of two 2-cycles and the 1-dimensional spaces fixed by  $x$  are in different orbits of  $y$ . We may suppose either  $xv_1 = \lambda v_5$ ,  $xv_2 = v_2$ ,  $xv_6 = v_6$  or  $xv_1 = \lambda v_6$ ,  $xv_2 = v_2$ ,  $xv_5 = v_5$ .

In the first case,  $\chi_z(t) = t^6 - (\lambda + \lambda^{-1})t^3 + 1$ . Comparing this polynomial with (3) we obtain  $\lambda^2 = -1$  and so  $(xy)^6 = -I$ , against Lemma 2.3. In the second case,  $\chi_z(t) = t^6 - \lambda t^4 - \lambda^{-1}t^2 + 1$ . Comparison with (3) gives  $c^2 - b^{q+1} = 2\gamma = 0$  and so

$p = 2$ . However in this case, we must have  $b + ac = 0$  that gives  $a^q = a^2$ , whence  $a^3 = 1$  and  $\det(J) = 0$ .

**Case (c)** The permutation induced by  $x$  is the product of two 2-cycles and the 1-dimensional spaces fixed by  $x$  are in the same orbit of  $y$ . We may suppose  $xv_1 = v_1$ ,  $xv_2 = v_2$ ,  $xv_5 = \lambda v_6$ ,  $xv_6 = \lambda^{-1}v_5$ . Consideration of the characteristic polynomial of  $z$  gives the conditions:  $b = 0$  (i.e.  $a^{2q} = 2a$ ),  $\lambda = a^q$  and  $\lambda^{-1} = a$ . It follows  $a^{q+1} = 1$  and  $2a^3 = 1$ . By the assumption  $a \notin \mathbb{F}_q$ , we have  $p \neq 2, 3$ . However  $1 = (2a^3)(2a^{3q})$  gives the absurd  $1 = 4$ .  $\square$

**Lemma 2.5.** *If  $\mathbb{F}_p[a^3] = \mathbb{F}_{q^2}$  then the projective image  $\overline{H}$  of  $H$  is not conjugate to any subgroup of  $\mathrm{PSL}_6(q_0)$  for any  $q_0 < q^2$ .*

*Proof.* Assume that  $\overline{H}$  is conjugate to a subgroup of  $\mathrm{PSL}_6(q_0)$ , i.e., that

$$g^{-1}Hg \leq \mathrm{SL}_6(\mathbb{F}_{q_0}) \langle -\omega I \rangle,$$

for some  $g$ . In particular  $g^{-1}zg\omega^{-j} \in \mathrm{SL}_6(\mathbb{F}_{q_0})$  for some  $j \in \{0, \pm 1\}$ . Set  $z_0 = g^{-1}zg\omega^{-j}$  and  $\chi_{z_0}(t) = \sum_{k=0}^6 f_k t^k$ . Clearly all  $f_k$ 's lie in  $\mathbb{F}_{q_0}$ . Then

$$\chi_z(t) = \chi_{g^{-1}zg}(t) = \chi_{z_0\omega^j}(t) = \sum_{k=0}^6 f_k \omega^{j(6-k)} t^k.$$

Comparing the coefficients of the terms of degrees 5 and 2 (see (3)), we get

$$-\frac{b+ac}{c} = \omega^j f_5, \quad -\frac{b}{c} = \omega^{4j} f_2 = \omega^j f_2.$$

It follows  $\omega^j(f_2 - f_5) = a$ , whence  $a^3 \in \mathbb{F}_{q_0}$ .  $\square$

**Lemma 2.6.** *If  $H$  is absolutely irreducible, then it is not conjugate to any maximal subgroup  $M$  with projective image  $\overline{M} = \mathrm{PSU}_2(q^2) \times \mathrm{PSU}_3(q^2)$ .*

*Proof.* The subgroup  $\overline{M}$  is contained in the projective image of  $\mathrm{GL}_2(\mathbb{F}) \otimes \mathrm{GL}_3(\mathbb{F})$ . Assume that our claim is wrong and write  $x = x_2 \otimes x_3$  with  $x_2 \in \mathrm{GL}_2(\mathbb{F})$  and  $x_3 \in \mathrm{GL}_3(\mathbb{F})$ . From  $x_2^2 \otimes x_3^2 = x^2 = I$ , we get  $x_2^2 = \rho I_2$  and  $x_3^2 = \rho^{-1} I_3$ . By the irreducibility of  $H$  neither  $x_2$  nor  $x_3$  can be scalar. So the rational canonical forms of  $x_2$  and  $x_3$  are respectively

$$\begin{pmatrix} 0 & \rho \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} \pm\sqrt{\rho^{-1}} & 0 & 0 \\ 0 & 0 & \rho^{-1} \\ 0 & 1 & 0 \end{pmatrix}.$$

It follows that the rational canonical form of  $x_2 \otimes x_3$  consists of 3 blocks  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ , in contrast with the rational canonical form of  $x$ .  $\square$

Let  $M$  be a maximal subgroup in class  $\mathcal{S}$ , whose projective image  $\overline{M}$  is isomorphic to  $\mathrm{PSU}_3(q^2)$ ,  $q$  odd. Denote by  $S^2(V)$  the symmetric square of  $V = \mathbb{F}^3$ . For  $\tau \in \{\mathrm{id}, \sigma\}$  consider the representation  $\varphi_\tau$ , of degree 6, defined by

$$\varphi_\tau(g) := (g^\tau \otimes g^\tau)_{|S^2(V)}, \quad \forall g \in \mathrm{SU}_3(q^2).$$

Since the groups in class  $\mathcal{S}$  are absolutely irreducible, by [3, 5.4.11] we have that, up to conjugation,  $M = \pm \varphi_\tau(\mathrm{SU}_3(q^2))$ , for some  $\tau \in \{\mathrm{id}, \sigma\}$ .

**Lemma 2.7.** *Assume  $q$  odd and  $H$  absolutely irreducible. Then  $H$  is not conjugate to any maximal subgroup  $M$  whose projective image is isomorphic to  $\mathrm{PSU}_3(q^2)$ .*

*Proof.* Assume the contrary. Then there exist  $\xi, \eta \in \text{SU}_3(q^2)$  such that  $X := \pm \varphi_\tau(\xi) = x^h$ ,  $Y := \varphi_\tau(\eta) = y^h$ , for some  $h$ . Thus  $\xi^2 = \pm I$  and  $\eta^3 = I$ . Since the space of fixed points of  $x$  has dimension 4, we need  $X := \varphi_\tau(\xi)$ . In particular  $\xi$  has a 2-dimensional eigenspace  $U$ . By the irreducibility we may assume  $U \cap \eta U = \langle e_2 \rangle$ . Setting  $e_1 = \eta^{-1}e_2$  we have  $e_1 \in U$  and  $\eta e_2 \notin U$ . Thus, up to conjugation:

$$\xi = \begin{pmatrix} -1 & 0 & s \\ 0 & -1 & r \\ 0 & 0 & 1 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Finally, for  $\xi\eta$  to be unitary, we need  $r = s^q$  (see [7]). With respect to the basis

$$e_1 \otimes e_1, e_2 \otimes e_2, e_3 \otimes e_3, \frac{e_1 \otimes e_2 + e_2 \otimes e_1}{2}, \frac{e_1 \otimes e_3 + e_3 \otimes e_1}{2}, \frac{e_2 \otimes e_3 + e_3 \otimes e_2}{2},$$

of  $S^2(V)$ , we get (e.g. assuming  $\tau = \text{id}$ ):

$$X = \begin{pmatrix} 1 & 0 & s^2 & 0 & -s & 0 \\ 0 & 1 & s^{2q} & 0 & 0 & -s^q \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2s^{q+1} & 1 & -s^q & -s \\ 0 & 0 & 2s & 0 & -1 & 0 \\ 0 & 0 & 2s^q & 0 & 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

The coefficients of  $\chi_{XY}(t)$  of degrees 1, 2, 3 are respectively:

$$(s^2 - s^q - a^q) - b^q/c; \quad (-s^{q+2} + s + s^{2q}) - b/c; \quad s^3 - 4s^{q+1}t + s^{3q} + 2.$$

Equating the coefficients of the terms  $t^i$  in  $\chi_{XY}(t)$  and  $\chi_{xy}(t)$  we get the following conditions  $P_i$ :

$$\begin{aligned} P_1 &= c(s^2 - s^q - a^q) - b^q = 0; & P_2 &= c(-s^{q+2} + s + s^{2q}) - b = 0; \\ P_3 &= s^3 - 4s^{q+1} + s^{3q} + 2 = 0. \end{aligned}$$

Here, we are considering  $b, c$  in the definition of  $y$  as free parameters, with the only restrictions  $b \in \mathbb{F}_{q^2}$ ,  $0 \neq c \in \mathbb{F}_q$ . Condition  $P_2 = 0$  gives  $b = c(-s^{q+2} + s + s^{2q})$ . Then  $P_1 = 0$  gives  $a = s^{q+2} - 2s$ . Now, recalling the definition of  $b$  and  $c$  as functions of  $a$  and substituting the previous expressions for  $a$  and  $b$ , we obtain

$$(s^{q+1} - 3)(s^{2q+2} - s^{3q} - 2s^{q+1} + 2) = 0.$$

If  $s^{q+1} = 3$ , then  $a = s$ . From  $P_3 = 0$  it follows  $a^{q+1} = 3$  and  $a^6 - 10a^3 + 27 = 0$ . Setting  $a^q = 3a^{-1}$ , we get  $\gamma = a^3 - 6a^{q+1} + a^{3q} + 8 = \frac{a^6 - 10a^3 + 27}{a^3} = 0$ , which implies  $H$  reducible.

So, assume  $P' = s^{2q+2} - s^{3q} - 2s^{q+1} + 2 = 0$ . In this case, the resultant between  $P_3$  and  $P'$ , with respect to  $s^q$ , is  $(s^3 - 1)(s^3 - 4)^3$ . If  $s^3 = 1$ , from  $P_3 = 0$  we get  $s^{q+1} = 1$ , whence  $a = -s$ . This means  $a = -\omega^i$  and  $a^q = -\omega^{2i}$  with  $i = 1, 2$ . However, for these values  $H$  is reducible ( $\det(J) = 0$ ). If  $s^3 = 4$ , from  $P_3 = 0$  we get  $2s^{q+1} = 5$ . Taking the third power of both sides,  $8(s^3)^q s^3 = 125$ , whence  $128 = 125$ , i.e.  $p = 3$ . This implies  $s^3 = 1$ , case that we have already excluded. Similarly, we can deal with the case  $\tau = \sigma$ .  $\square$

**Lemma 2.8.** *Assume  $H$  absolutely irreducible. Then  $H$  is not contained in any maximal subgroup  $M$  in class  $\mathcal{S}$ .*

*Proof.* We refer to [1, Table 8.27, page 391]. Case by case, we exclude that  $H$  is contained in  $M$  with projective image  $\overline{M}$ .

**Case (a)**  $\overline{M} = \text{PSU}_3(q^2)$ ,  $q$  odd. By Lemma 2.7 this cannot occur.

From now on we may assume  $q = p \geq 3$ .

**Case (b)**  $\overline{M} \in \{\text{Alt}(6), \text{Alt}(6).2_3, \text{Alt}(7), \text{PSL}_3(4), \text{PSL}_3(4).2_1\}$ . Simply observe that the only possible values for the order of the projective image of  $xy$  are  $\leq 8$ , which is excluded by Lemma 2.3.

**Case (c)**  $M = Z(\text{SU}(6, q^2)) \circ \text{SL}_2(11)$ ,  $2 < p \equiv 2, 6, 7, 8, 10 \pmod{11}$ . Notice that the only involution in  $\text{SL}_2(11)$  is  $-I$ . Thus, if  $H \leq M$ , it follows  $x \in Z(M)$  and so  $x$  should commute with  $y$ , but this does not happen.

**Case (d)**  $M = 6_1.\text{PSU}_4(9)$ ,  $p \equiv 5 \pmod{12}$ .

Considering the eigenvalues of an irreducible representation of degree 6 of  $M$ , we have that  $x$  belongs to the class  $2c$  and  $y \in \{3i, 3j\}$  (in GAP notation). We determine the possible orders of  $xy$  calculating the structure constants using the character table of  $M$ . Notice that  $xy$  cannot have eigenvalues of multiplicity greater than 1.

If  $y \in 3i$ , then  $xy \in \{9g, 9h\}$ . However, in this case  $(xy)^9$  is scalar. If  $y \in 3j$ , then  $xy \in \{7a, 7b, 8a, 8b, 9g, 9h, 14a, 14b, 21a, 21b, 21c, 21d, 24a, 24b, 24c, 24d, 42a, 42b, 42c, 42d\}$ . However, for any choice of  $xy$  we obtain a contradiction with Lemma 2.3, except when  $xy \in \{24a, 24b, 24c, 24d\}$ . In these last cases the characteristic polynomial of  $xy$  is  $t^6 + \omega^{2j}t^4 + \omega^j t^2 + 1$  ( $j = 1, 2$ ). Comparison with (3) gives  $b + ac = b + c\omega^j = 0$ , whence  $a = \omega^j$ . Since  $a \notin \mathbb{F}_q$ ,  $a^q = \omega^{2j}$  and so  $b + ac = -2\omega^j$ , a contradiction.

**Case (e)**  $M = 6_1.\text{PSU}_4(9).2_2$ ,  $p \equiv 11 \pmod{12}$ .

Considering the eigenvalues of an irreducible representation of degree 6 of  $M$ , we have  $x \in 2c$  and  $y \in \{3i, 3j\}$ . We proceed as done in item (1). If  $y \in 3i$ , then  $xy$  must belong to class  $9g$ , a contradiction with Lemma 2.3. If  $y \in 3j$ , then  $xy \in \{7a, 8a, 9g, 14a, 21a, 21b, 24a, 24b, 42a, 42b\}$ . However, in each case we obtain a contradiction with Lemma 2.3, except when  $xy \in \{24a, 24b\}$ . As before, in these last cases the characteristic polynomial of  $xy$  is  $t^6 + \omega^{2j}t^4 + \omega^j t^2 + 1$  ( $j = 1, 2$ ), which leads to a contradiction.  $\square$

It can be shown that the group  $\text{SU}_6(4)$  cannot be generated by an element  $x$  having similarity invariants  $(t-1), (t-1), (t^2-1), (t^2-1)$  and an element  $y$  of order 3. However, choosing  $x$  with similarity invariants  $(t^2-1), (t^2-1), (t^2-1)$  we get the following:

**Proposition 2.9.** *The group  $\text{SU}_6(4)$  is (2, 3)-generated.*

*Proof.* Take the following two matrices of  $\text{SL}_6(4)$ :

$$(5) \quad x = x_6 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & \omega \\ 0 & 0 & 1 & 0 & \omega^2 & \omega \\ 0 & 0 & 0 & \omega & 1 & \omega^2 \\ 0 & 0 & \omega^2 & \omega^2 & \omega & 0 \end{pmatrix}, \quad y = y_6 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then  $x^2 = y^3 = 1$ . Moreover  $x^T x^\sigma = y^T y^\sigma = I$  and so  $H = \langle x, y \rangle \leq \text{SU}_6(4)$ . Notice that  $xy$ ,  $[x, y]$  and  $([x, y]^2 xy)$  have order 11, 9 and 21, respectively. The only maximal subgroups  $M$  of  $\text{SU}_6(4)$  whose orders are divisible by  $7 \cdot 9 \cdot 11$  are of

type  $3 \cdot M_{22}$ . On the other hand,  $3 \cdot M_{22}$  does not have elements of order 9, whence  $H = \text{SU}_6(4)$ .  $\square$

**Theorem 2.10.** *Let  $x, y$  as in (1), with  $a \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ , such that  $a^{q+1} \neq 4$  and*

- (i)  $a^3 - 6a^{q+1} + a^{3q} + 8 \neq 0$ ;
- (ii)  $a^{2q+2} - 5a^{q+1} + 8 \neq 0$ ;
- (iii)  $\mathbb{F}_p[a^3] = \mathbb{F}_{q^2}$ .

*Then  $H = \langle x, y \rangle = \text{SU}_6(q^2)$ . Moreover, if  $q^2 \neq 2^2$ , then there exists  $a \in \mathbb{F}_{q^2}^*$  satisfying conditions (i) to (iii).*

*Proof.* By Proposition 2.9 we may assume  $q > 2$ . Conditions (i) and (ii) imply that  $H$  is absolutely irreducible (Theorem 2.2). By Lemma 2.1 we have  $H \leq \text{SU}_6(q^2)$ . Let  $M$  be a maximal subgroup of  $\text{SU}_6(q^2)$  which contains  $H$ . As observed before, Conditions (i) and (ii) imply that  $M \notin \mathcal{C}_1 \cup \mathcal{C}_3$ . Moreover  $M \notin \mathcal{C}_2$  by Lemma 2.4. From Condition (iii) and Lemma 2.5 we get  $M \notin \mathcal{C}_5$ . From Lemma 2.6 it follows  $M \notin \mathcal{C}_4$ . Finally, the case  $M \in \mathcal{S}$  is excluded by Lemmas 2.7 and 2.8.

We now prove that, for  $q > 2$ , there exists  $a \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$ , with  $a^{q+1} \neq 4$ , satisfying Conditions (i) to (iii). Take an element  $a \in \mathbb{F}_{q^2}^*$  of order  $q^2 - 1$ . Then  $a^{q+1} \neq 4$ . Indeed assume  $a^{q+1} = 4$ : clearly  $p \neq 2$  and so  $a^{\frac{q+1}{2}} = \pm 2$ , whence the contradiction  $a^{\frac{q^2-1}{2}} = 1$ . By Lemma 2.4. of [7], we have  $\mathbb{F}_p[a^3] = \mathbb{F}_{q^2}$ , since we are assuming  $q > 2$ . Now, if  $q \geq 23$  there are at least  $5q + 3$  elements of  $\mathbb{F}_{q^2}$  of order  $q^2 - 1$  (use [7, Lemma 2.1] for  $q \geq 127$  and direct computations otherwise). Thus, our claim is proved since the elements which do not satisfy (i) or (ii) are at most  $5q + 2$ . For  $2 < q \leq 19$  take  $a$  whose minimal polynomial  $m_a(t)$  over  $\mathbb{F}_p$  is given below.

| $q$       | $m_a(t)$      | $q$  | $m_a(t)$        | $q$   | $m_a(t)$                  |
|-----------|---------------|------|-----------------|-------|---------------------------|
| 3, 13     | $t^2 + t + 2$ | 4, 9 | $t^4 + t^3 - 1$ | 5, 11 | $t^2 + t - 3$             |
| 7, 17, 19 | $t^2 + t + 3$ | 8    | $t^6 + t + 1$   | 16    | $t^8 + t^6 + t^5 + t + 1$ |

Then  $a$  satisfies (i), (ii) and (iii) and hence  $H = \text{SU}_6(q^2)$ , for all  $q > 2$ .  $\square$

### 3. TOWARDS A UNIFORM (2, 3)-GENERATION OF $\text{SU}_n(q^2)$ , $n \geq 6$ .

Write  $n = 3m + r$  with  $0 \leq r \leq 2$ . Let  $V$  be the vector space over  $\mathbb{F}_{q^2}$  with basis:

$$(6) \quad B = \{e_i \mid 1 \leq i \leq 3m\}, \text{ if } r = 0, \quad \widehat{B} = \{v_i \mid 1 \leq i \leq r\} \cup B, \text{ if } r > 0.$$

For  $n > 6$ , we consider the following elements of  $\text{SL}_n(q^2)$ :

$$(7) \quad x_n = \nu_n x_6, \quad y_n = \mu_n y_6,$$

where  $x_6$  and  $y_6$  act on  $\langle e_{n-j} \mid 0 \leq j \leq 5 \rangle$  as in (1) if  $q > 2$ , as in (5) if  $q = 2$ . The remaining basis vectors are fixed by  $x_6, y_6$ .

The matrices  $\nu_n$  and  $\mu_n$ ,  $n \geq 7$ , are monomial and act as follows.

$$(8) \quad q > 2: \quad \nu_n = \eta_1 \eta_2 \prod_{k=1}^{m-2} (e_{3k}, e_{3k+1}), \quad \mu_n = \prod_{k=0}^{m-3} (e_{3k+1}, e_{3k+2}, e_{3k+3})$$

where  $\eta_1 = \text{id}$  if  $r = 0$ ,  $\eta_1 = \prod_{i=1}^r (v_i, e_i)$  if  $r > 0$ ;  $\eta_2 = \text{id}$  when  $q$  is even or  $n = 8$ ,  $\eta_2(e_{3m-4}) = \pm e_{3m-4}$ , so that  $\det(x_n) = 1$ , otherwise.



For  $q = 2$ ,  $n \neq 7, 8$ :

$$\nu_n = \theta \prod_{k=1}^{m-3} (e_{3k}, e_{3k+1}) (e_{3m-6}, e_{3m-5}, e_{3m-4}), \quad \mu_n = \delta \prod_{k=0}^{m-3} (e_{3k+1}, e_{3k+2}, e_{3k+3}),$$

where  $\theta = (e_1, e_2)$  if  $r = 0$ ,  $\theta = (e_1, v_1)$  if  $r = 1$ ,  $\theta = (e_1, v_1)(e_2, v_2)$  if  $r = 2$ ;  $\delta = \text{id}$  if  $r = 0, 1$  and  $\delta(v_i) = \omega^i v_i$  if  $r = 2$ .

For  $n = 7, 8$ ,  $\nu_7 = (v_1, e_1, e_2)$ ,  $\nu_8 = (v_1, e_1, v_2, e_2)$ .

It is easy to see that  $x_n$  and  $y_n$  have respective orders 2, 3 and belong to  $\text{SU}_n(q^2)$ . Computer evidences, combined with the permutational methods of [11], [13] and [14], strongly suggest that  $\langle x_n, y_n \rangle = \text{SU}_n(q^2)$  whenever  $\langle x_6, y_6 \rangle = \text{SU}_6(q^2)$ . The cases  $n = 8$ ,  $q = 3, 4$  require a slight modification of the action of  $y$  on  $\langle v_1, v_2 \rangle$ .

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DIPARTIMENTO DI MATEMATICA E FISICA, UNIVERSITÀ CATTOLICA DEL SACRO CUORE, VIA MUSEI 41, I-25121 BRESCIA, ITALY

*E-mail address:* marcoantonio.pellegrini@unicatt.it

*E-mail address:* mariateresa.prandelli@istruzione.it

DIPARTIMENTO DI MATEMATICA E FISICA, UNIVERSITÀ CATTOLICA DEL SACRO CUORE, VIA MUSEI 41, I-25121 BRESCIA, ITALY

*E-mail address:* mariaclara.tamburini@unicatt.it